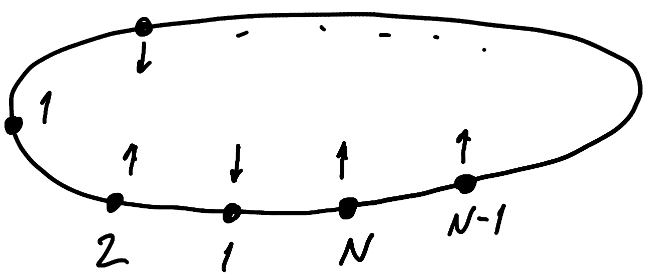
1D Ising model and transfer matrices

Focus on the 1D Ising model

$$H = -J\sum_{(ij)} s_i s_j - h\sum_i s_i$$

Since the boundary conditions should not matter in the thermodynamic limit $(N \to \infty)$, choose periodic boundary conditions.



$$Z = \sum_{\{s\}} e^{\beta J(s_1 s_2 + s_2 s_3 + \ldots + s_N s_1)}\, e^{\beta h(s_1 + s_2 + \ldots + s_N)} =$$

$$= \sum_{\{s\}} e^{\beta J s_1 s_2 + \frac{\beta h}{2}(s_1+s_2)}\, e^{\beta J s_2 s_3 + \frac{\beta h}{2}(s_2+s_3)} \ldots e^{\beta J s_N s_1 + \frac{\beta h}{2}(s_1+s_N)}$$

$$= \sum_{\{s\}} \langle s_1|\hat{T}|s_2\rangle\langle s_2|\hat{T}|s_3\rangle \ldots \langle s_N|\hat{T}|s_1\rangle$$

where $\langle s_i|\hat{T}|s_{i+1}\rangle = e^{\beta J s_i s_{i+1} + \frac{\beta h}{2}(s_i + s_{i+1})}$

Because s_i and s_{i+1} can take only two values, ± 1, we may introduce a 2×2 matrix

$$T = \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix}$$ – transfer matrix

Then $Z = \mathrm{Tr}\, T^N = \lambda_1^N + \lambda_2^N$, where –

Then $Z = \mathrm{Tr}\, T^N = \lambda_+^N + \lambda_-^N$, where λ_+ and λ_- are the eigenvalues of matrix T

$$0 = \det \begin{vmatrix} \lambda - e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & \lambda - e^{\beta(J-h)} \end{vmatrix} = \lambda^2 - 2\lambda e^{\beta J}\cosh(\beta h) + 2\sinh(2\beta J)$$

$$\frac{\mathcal{D}}{2} = e^{2\beta J}\cosh^2(\beta h) - 2\sinh(2\beta J) = e^{2\beta J}\sinh^2(\beta h) + e^{-2\beta J}$$

$$\boxed{\lambda_\pm = e^{\beta J}\left[\cosh(\beta h) \pm \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\,\right]}$$

In the thermodynamic limit $(N \to \infty)$, $\lambda_+^N + \lambda_-^N \to \lambda_+^N$

The free energy

$$\boxed{F = -NT\ln\left(e^{\beta J}\left[\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\,\right]\right)}$$

Magnetic susceptibility

Magnetisation may be defined as

$$m = -\frac{\partial F}{\partial h}$$

$$-\frac{\partial F}{\partial h} = T\frac{1}{Z}\frac{\partial Z}{\partial h} = \frac{\mathrm{Tr}\left(\sum_i s_i e^{-\beta H}\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)} \equiv \left\langle \sum_i s_i \right\rangle$$

(So, $m = \langle \sum_i s_i \rangle$ may also be used as a definition)

$$m = -\frac{\partial F}{\partial h} = N\,\frac{\sinh(\beta h) + \dfrac{\cosh(\beta h)\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}}{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}$$

$$\boxed{m = N\frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h)+e^{-4\beta J}}}}$$

No singularity!
(no phase transition)

As $h \to 0$ $\quad m \to 0$

$$\chi(h=0) = \left.\frac{\partial m}{\partial h}\right|_{h\to 0} = \frac{1}{T} e^{\frac{2J}{T}}$$ - susceptibility

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Heat capacity

set $h = 0$

$$S = -\frac{\partial F}{\partial T} = \beta^2 \frac{\partial F}{\partial \beta}$$

$$S = N \ln 2 + N \ln[\cosh(\beta J)] - N\frac{J}{T}\tanh(\beta J)$$

$$\boxed{C = N\frac{J^2}{T^2}\frac{1}{\cosh^2(\beta J)}}$$

No discontinuity

Correlation length

Define $G(r) = \langle S_k S_{k+r}\rangle - \langle S_k\rangle\langle S_{k+r}\rangle$

Translational invariance (in the thermodynamic limit)

It is expected in disordered phases that

$$G(r) \propto e^{-\frac{r}{\xi}}, \quad \xi \text{ - correlation length}$$

$$G(r) = \langle S_1 S_{r+1}\rangle = \frac{1}{Z}\sum_{\{S\}} S_1 e^{-\beta H} S_{r+1} =$$

$$= \frac{1}{Z}\sum S_1 \langle S_1|\hat{T}|S_2\rangle\langle S_2|\hat{T}|S_3\rangle \ldots \langle S_r|\hat{T}|S_{r+1}\rangle S_{r+1} \times$$

$$= \frac{1}{Z} \sum_{\{s\}} s_1 \langle s_1 | \hat{T} | s_2 \rangle \langle s_2 | \hat{T} | s_3 \rangle \dots \langle s_r | \hat{T} | s_{r+1} \rangle s_{r+1} \times$$
$$\times \langle s_{r+1} | \hat{T} | s_{r+2} \rangle \dots \langle s_N | \hat{T} | s_1 \rangle \equiv$$

Using the previously found decomposition

$$\equiv \frac{1}{Z} \mathrm{Tr} \left(\hat{s}_1 \hat{T}^r \hat{s}_{r+1} \hat{T}^{N-r} \right) =$$

$$= \frac{1}{Z} \sum_{\mu, \eta} \langle \mu | \hat{\sigma}_z | \eta \rangle \eta^r \langle \eta | \hat{\sigma}_z | \mu \rangle \mu^{N-r} \quad (1)$$

The dominant in the thermodynamic limit contribution comes from the eigenvalue $\mu = \lambda_+$

The eigenvalues with eigenvectors:

$$\lambda_+ = 2\cosh(\beta J) \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_- = 2\sinh(\beta J) \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\langle \lambda_+ | \hat{\sigma}_z | \lambda_+ \rangle = \langle \lambda_- | \hat{\sigma}_z | \lambda_- \rangle = 0$$

$$\langle \lambda_+ | \hat{\sigma}_z | \lambda_- \rangle = \langle \lambda_- | \hat{\sigma}_z | \lambda_+ \rangle = 1$$

Recall that $Z \approx \lambda_+^N$

We've demonstrated that $\mu = \lambda_+$, $\eta = \lambda_-$

Then from (1)

$$G(r) = \left(\frac{\lambda_-}{\lambda_+} \right)^r = \left(\tanh(\beta J) \right)^r = e^{r \ln \tanh(\beta J)}$$

$$\boxed{\xi^{-1} = -\ln \tanh(\beta J)}$$

At $T \to 0$ $\quad \xi \to \infty$

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$(T \gg J)$